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Jacobson radical algebras with quadratic growth

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Abstract

In this paper, it is shown that over every countable algebraically closed field $\mathbb{K}$ there exists a finitely generated $\mathbb{K}$-algebra that is Jacobson radical, infinite dimensional, generated by two elements, graded, and has quadratic growth. We also propose a way of constructing examples of algebras with quadratic growth that satisfy special types of relations.

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Introduction

Algebras with linear growth were described by Small, Stafford and Warfield in [6]. In [3] (pp. 18) Bergman proved that algebras with growth function smaller than $f(n) = \frac{n(n+1)}{2}$ have linear growth. What properties would algebras with a growth function close to $f(n) = \frac{n(n+1)}{2}$ satisfy? Examples of primitive algebras with very small growth functions were constructed by

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Usi Vishne using Moorse trajectories [9]. In [1] Bartholdi constructed self-similar algebras with very small growth functions over the field $\mathbb{F}_2$ which are graded nil. In fact, all algebras constructed in [1] are primitive and hence not Jacobson radical (as mentioned in [8]).

We will construct an example with growth function bounded above by $n^2 + 4n + 3$ which are both infinite dimensional and Jacobson radical. It is unclear whether this algebra is nil. We will also present a way to construct other examples which are bounded above by the same growth function.

Recall that non-nil Jacobson radical algebras with Gelfand-Kirillov dimension two were constructed in [8], and nil algebras with Gelfand-Kirillov dimension not exceeding three were constructed in [5]. It is not known if there are nil algebras with quadratic growth, or more generally with Gelfand-Kirillov dimension two.

Our first main result is the following:

Theorem 0.1. *Let $\mathbb{K}$ be an algebraically closed field. Let $A = \mathbb{K}\langle x, y\rangle$ to be the free noncommutative algebra generated (in degree one) by the elements x, y. Let $H(n) \subset A$ be the homogeneous subspace of degree $n \geq 0$. Finally, for any $F \subseteq H(n)$, let:*

$$\mathcal{E}(F) = \bigcap_{j=0}^{n-1} \sum_{k=0}^{\infty} H(kn + j)FA.$$

For any sequence $\{N_i\}_{i\in\mathbb{N}}$ of strictly increasing natural numbers and any sequence $\{F_i\}_{i\in\mathbb{N}}$ of homogeneous subspaces such that $F_i \subseteq H(2^{N_i})$ and $\dim F_i < \frac{1}{2}(N_i - N_{i-1} + 1)$, the quotient algebra $A/\langle\mathcal{E}(F_i)\rangle_{i\in\mathbb{N}}$ can be homomorphically mapped onto an infinite dimensional graded algebra with quadratic or linear growth. Moreover, the dimension of this algebra's homogeneous subspace of dimension n would be bounded above by $2n + 2$.

In other words, there's a graded ideal $E \lhd A$ such that $\bigcup_{i\in\mathbb{N}} \mathcal{E}(F_i) \subseteq E$ and A/E is infinite dimensional and has quadratic growth. Specifically, $1 \leq H(n)/(E \cap H(n)) \leq 2n+2$ for each $n \geq 1$. As a corollary we get the following result.

Corollary 0.2. *Over every countable, algebraically closed field $\mathbb{K}$ there exists a finitely generated $\mathbb{K}$ algebra that's Jacobson radical, infinite dimensional, generated by two elements, graded and has quadratic growth.*

We also propose a new way of constructing examples of algebras with quadratic growth satisfying special types of relations.

The general path of the proof is as follows:

- Subspaces $U(2^n), V(2^n) \subseteq H(2^n)$ are constructed, depending on $U(2^i)$, $V(2^i)$ for $i < n$. This part bears resemblance to results from [4]. Properties that the $V(2^n)$ spaces exhibit include $V(2^{n-1})^2 \subseteq V(2^n)$ and $\dim V(2^n) = 2$, the latter being instrumental in establishing quadratic growth. We assure that sets $\{F_i\}_{i\in\mathbb{N}}$ are contained in our sets $U(2^n)$.

- In section 3 we introduce ideal E, whose construction uses the sets $U(2^n)$, in order to arrive at our desired quotient A/E. Note that the ideal E is defined differently than in [4]. We then find an upper bound of the growth of A/E.

- In sections 4 and 5 we show that for some appropriate choice of sets $\{F_i\}$, the constructed algebra A/E is Jacobson radical.

We wrap up the proof of Theorem and its corollary in section 5.

1 Notation

In what follows, $\mathbb{K}$ is a countable field and $A = \mathbb{K}\langle x, y\rangle$ is the free $\mathbb{K}$-algebra in two non-commuting indeterminates x and y. The monomials in this algebra will be the products of the form $x_1 \cdots x_n$, with each $x_i \in \{x, y\}$ (whereas the monomials *with coefficient* will be of the form $kx_1 \cdots x_n$ with $k \in \mathbb{K}$). The degree of a monomial is the length of this product. For any $n \geq 0$, $H(n)$ will denote the homogeneous subspace of degree n: the $\mathbb{K}$-space generated by the degree-n monomials. Finally, $\bar{A} = \sum_{n=1}^{\infty} H(n)$ will be the $\mathbb{K}$-space of polynomials with no constant term.

2 Constructing sets $U(2^n)$ and $V(2^n)$

Suppose we have a strictly increasing sequence of naturals $\{N_i\}_{i=0}^{\infty}$ with $N_0 = 1$ and a sequence of homogeneous subspaces $\{F_i\}_{i=0}^{\infty}$ with each $F_i \subseteq 2^{N_i}$ and $F_0 = (0)$.

In this section, we address the question: does there exist, for every $i \geq 0$, a subspace $U_i \subset H(2^i)$ and two monomials (with non-zero coefficient) $v_{i,1}, v_{i,2} \in H(2^i)$ such that, for each $i \geq 0$:

1. $U_i \oplus \mathbb{K}v_{i,1} \oplus \mathbb{K}v_{i,2} = H(2^i)$.
2. There exists a $v \in \mathbb{K}v_{i,1} + \mathbb{K}v_{i,2}$ such that $U_{i+1} = H(2^i)U_i + U_iH(2^i) + vH(2^i)$.
3. $F_i \subseteq U_{N_i}$.

We will eventually set $V_i = \mathbb{K}v_{i,1} + \mathbb{K}v_{i,2}$, so that $U_i \oplus V_i = H(2^i)$.

We shall attack the problem with induction. For the base case, set U_0 as an arbitrary subspace of $H(1)$ with $\dim U_0 = \dim H(1) - 2$, and set $v_{0,1}$, $v_{0,2}$ as two linearly independent monomials such that $U_0 + \mathbb{K}v_{0,1} + \mathbb{K}v_{0,2} = H(1)$.

For the inductive step, assume the existence of $U_{N_i}, v_{N_i,1}, v_{N_i,2}$ for some $i \geq 0$, and find possible $U_k, v_{k,1}, v_{k,2}$ for all $N_i < k \leq N_{i+1}$.

Let $W \cong \mathbb{K}^{2(N_{i+1}-N_i)}$ be a $\mathbb{K}$-space with indices $\{x_{k,1}, x_{k,2}\}_{k=N_i}^{N_{i+1}-1}$, let W_k be the subspace of all elements where $(x_{k,1}, x_{k,2}) = (0,0)$, and let $\overline{W} = W \setminus \bigcup_{k=N_i}^{N_{i+1}-1} W_k$.

Given some vector $\vec{w} \in \overline{W}$, define $U_k(\vec{w}), v_{k,1}(\vec{w}), v_{k,2}(\vec{w})$ recursively for each $N_i \leq k \leq N_{i+1}$, as follows: first, set $U_{N_i}(\vec{w}) = U_{N_i}$, $v_{N_i,1}(\vec{w}) = v_{N_i,1}$, $v_{N_i,2}(\vec{w}) = v_{N_i,2}$.

Then, assuming $U_k(\vec{w}), v_{k,1}(\vec{w}), v_{k,2}(\vec{w})$ are defined for some $N_i \leq k < N_{i+1}$:

$$U_{k+1}(\vec{w}) = H(2^k)U_k(\vec{w}) + U_k(\vec{w})H(2^k) + (x_{k,2}(\vec{w})v_{k,1}(\vec{w}) - x_{k,1}(\vec{w})v_{k,2}(\vec{w}))H(2^k).$$

If $x_{k,1}(\vec{w}) \neq 0$, set:

$$v_{k+1,1}(\vec{w}) = x_{k,1}(\vec{w})^{-1}v_{k,1}^2(\vec{w}),$$

$$v_{k+1,2}(\vec{w}) = x_{k,1}(\vec{w})^{-1}v_{k,1}(\vec{w})v_{k,2}(\vec{w}),$$

and if $x_{k,1}(\vec{w}) = 0$, then $x_{k,2}(\vec{w}) \neq 0$, so set:

$$v_{k+1,1}(\vec{w}) = x_{k,2}(\vec{w})^{-1}v_{k,2}(\vec{w})v_{k,1}(\vec{w}),$$

$$v_{k+1,2}(\vec{w}) = x_{k,2}(\vec{w})^{-1}v_{k,2}^2(\vec{w}).$$

For any $\vec{w} \in \overline{W}$, this clearly satisfies conditions (1-2).

Lemma 2.1. *For any $N_i \leq k < N_{i+1}$, $a, b \in \{1, 2\}$, $\vec{w} \in \overline{W}$,*

$$v_{k,a}(\vec{w})v_{k,b}(\vec{w}) \in x_{k,a}(\vec{w})v_{k+1,b}(\vec{w}) + U_{k+1}(\vec{w})$$

Proof. If $x_{k,1}(\vec{w}) \neq 0$, and $a = 1$, $v_{k,a}(\vec{w})v_{k,b}(\vec{w}) = x_{k,a}(\vec{w})v_{k+1,b}(\vec{w})$.

If $x_{k,1}(\vec{w}) \neq 0$, and $a = 2$,

$$v_{k,a}(\vec{w})v_{k,b}(\vec{w}) = x_{k,a}(\vec{w})v_{k+1,b}(\vec{w}) + x_{k,1}(\vec{w})^{-1}(x_{k,2}(\vec{w})v_{k,1}(\vec{w}) - x_{k,1}(\vec{w})v_{k,2}(\vec{w}))v_{k,b}(\vec{w}).$$

If $x_{k,1}(\vec{w}) = 0$ and $a = 1$,

$$v_{k,a}(\vec{w})v_{k,b}(\vec{w}) = x_{k,2}(\vec{w})^{-1}(x_{k,2}(\vec{w})v_{k,1}(\vec{w}) - x_{k,1}(\vec{w})v_{k,2}(\vec{w}))v_{k,b}(\vec{w}).$$

And if $x_{k,1}(\vec{w}) = 0$ and $a = 2$, $v_{k,a}(\vec{w})v_{k,b}(\vec{w}) = x_{k,2}(\vec{w})v_{k+1,b}(\vec{w})$. $\square$

Let $P = \mathbb{K}[x_{k,1}, x_{k,2}]_{k=N_i}^{N_{i+1}-1}$, i.e. the (commutative) algebra of polynomial functions $W \to \mathbb{K}$. Let $Q = \prod_{k=N_i}^{N_{i+1}-1}(\mathbb{K}x_{k,1} + \mathbb{K}x_{k,2})^{2^{N_{i+1}-k-1}}$ be a homogenous subspace of P.

Theorem 2.2. *For any sequence $\{s_k\}_{k=1}^{2^{N_{i+1}-N_i}}$ of $\{1, 2\}$, there exists some $p_s \in Q$ such that for any $\vec{w} \in \overline{W}$,*

$$\prod_{k=1}^{2^{N_{i+1}-N_i}} v_{N_i,s_k} \in p_s(\vec{w})v_{N_{i+1},s_{2^{N_{i+1}-N_i}}}(\vec{w}) + U_{N_{i+1}}(\vec{w}).$$

Proof. We will use induction to show that, for any $0 \leq h \leq N_{i+1} - N_i$ and any sequence $\{s_k\}_{k=1}^{2^h}$ of $\{1, 2\}$,

$$\prod_{k=1}^{2^h} v_{N_i,s_k} \in \left(\prod_{j=0}^{h-1}\prod_{k=1}^{2^{h-j-1}} x_{N_i+j,s_{2^j(2k-1)}}(\vec{w})\right) v_{N_i+h,s_{2^h}}(\vec{w}) + U_{N_i+h}(\vec{w}),$$

with the end result of the theorem proven when $h = N_{i+1} - N_i$.

The base case is simply $v_{N_i,s_1} \in v_{N_i,s_1}(\vec{w}) + U_{N_i}(\vec{w})$.

For the inductive step, let $\{s_k\}_{k=1}^{2^{h+1}}$ be a sequence of $\{1,2\}$, and assume the inductive statement is true for $\{s_k\}_{k=1}^{2^h}$ and $\{s_k\}_{k=2^h+1}^{2^{h+1}}$. Lemma 2.1 shows that:

$$v_{N_i+h,s_{2^h}}(\vec{w})v_{N_i+h,s_{2^{h+1}}}(\vec{w}) \in x_{N_i+h,s_{2^h}}(\vec{w})v_{N_i+h+1,s_{2^{h+1}}}(\vec{w}) + U_{N_i+h+1}(\vec{w}).$$

Therefore,

$$\prod_{k=1}^{2^{h+1}} v_{N_i,s_k} \in \left(\left(\prod_{j=0}^{h-1}\prod_{k=1}^{2^{h-j-1}} x_{N_i+j,s_{2^j(2k-1)}}(\vec{w})\right) v_{N_i+h,s_{2^h}}(\vec{w}) + U_{N_i+h}(\vec{w})\right)\cdot$$

$$\left(\left(\prod_{j=0}^{h-1}\prod_{k=1}^{2^{h-j-1}} x_{N_i+j,s_{2^j(2k-1)+2^h}}(\vec{w})\right) v_{N_i+h,s_{2^{h+1}}}(\vec{w}) + U_{N_i+h}(\vec{w})\right) \subseteq$$

$$\left(\prod_{j=0}^{h-1}\prod_{k=1}^{2^{h-j}} x_{N_i+j,s_{2^j(2k-1)}}(\vec{w})\right) x_{N_i+h,s_{2^h}}(\vec{w})v_{N_i+h+1,s_{2^{h+1}}}(\vec{w}) + U_{N_i+h+1}(\vec{w}) =$$

$$\left(\prod_{j=0}^{h}\prod_{k=1}^{2^{h-j}} x_{N_i+j,s_{2^j(2k-1)}}(\vec{w})\right) v_{N_i+h+1,s_{2^{h+1}}}(\vec{w}) + U_{N_i+h+1}(\vec{w}).$$

$\square$

Corollary 2.3. *For any $f \in H(2^{N_{i+1}})$, there exists $p, q \in Q$ such that $\forall \vec{w} \in \overline{W}$, $f \in p(\vec{w})v_{N_{i+1},1}(\vec{w}) + q(\vec{w})v_{N_{i+1},2}(\vec{w}) + U_{N_{i+1}}(\vec{w})$.*

Proof. First, note that:

$$H(2^{N_{i+1}}) = (U_{N_i} + \mathbb{K}v_{N_i,1} + \mathbb{K}v_{N_i,2})^{2^{N_{i+1}-N_i}} =$$

$$(\mathbb{K}v_{N_i,1} + \mathbb{K}v_{N_i,2})^{2^{N_{i+1}-N_i}} + \sum_{k=1}^{2^{N_{i+1}-N_i}} H((k-1)2^{N_i})U_{N_i}H(2^{N_{i+1}} - k2^{N_i})$$

And for each $f \in H(2^{N_{i+1}})$, there exists a $f' \in (\mathbb{K}v_{N_i,1} + \mathbb{K}v_{N_i,2})^{2^{N_{i+1}-N_i}}$ such that, for any $\vec{w} \in \overline{W}$, $f \in f' + U_{N_{i+1}}(\vec{w})$.

Since f' can be written as a linear combination of the elements of the form $\prod_{k=1}^{2^{N_{i+1}}} v_{N_i,s_k}$, it's sufficient to prove the corollary over these elements, which is done in theorem 2.2. □

Let $d = \dim F_{i+1}$, let $\{f_k\}_{k=1}^d$ be elements that generate F_{i+1}, and let $\{p_k, q_k\} \subseteq Q$ be such that $\forall \vec{w} \in \overline{W}$, $f_k \in p_k(\vec{w})v_{N_{i+1},1}(\vec{w}) + q_k(\vec{w})v_{N_{i+1},2}(\vec{w}) + U_{N_{i+1}}(\vec{w})$, as detailed in corollary 2.3. If there exists a $\vec{w} \in \overline{W}$ such that each $p_k(\vec{w}) = q_k(\vec{w}) = 0$, then we can set $(U_k, v_{k,1}, v_{k,2}) = (U_k(\vec{w}), v_{k,1}(\vec{w}), v_{k,2}(\vec{w}))$, and condition (4) can be satisfied.

Let $G = \sum_{k=1}^d \mathbb{K}p_k + \mathbb{K}q_k \subseteq Q$ be the vector space generated by $\{p_k, q_k\}$. Our remaining goal is to show $\exists \vec{w} \in \overline{W} : G(\vec{w}) = (0)$.

Let R be the algebra generated by Q, i.e. $R = \sum_{k=1}^\infty Q^k$.

Lemma 2.4. *If G, P are defined as above, then:*

$$R \cap GP \subseteq G + GR.$$

Proof. Let M be the set of all monomials of P (without coefficient). Let M_Q be the monomials that generate Q, let $M_R = \bigcup_{j=1}^\infty M_Q^j$ be the monomials that generate R, and let $M'_R = M \backslash (M_R \cup \{1\})$. P can be decomposed: $P = \mathbb{K} \oplus R \oplus \mathbb{K}M'_R$.

Note that for any $m \in M_Q$ and any $m' \in M'_R$, $mm' \in M'_R$. As R is generated by monomials, $R \cap QM'_R = (0)$.

Let $g \in G$, and let $p \in P$ have the decomposition $p = k + r + s$, with $k \in \mathbb{K}$, $r \in R$ and $s \in \mathbb{K}M'_R$. Suppose that $gp \in R$. Since $gk + gr \in R$, $gs \in R \cap QM'_R = (0)$. Therefore, $gp \in \mathbb{K}g + gR$, and $R \cap GP \subseteq G + GR$. □

Theorem 2.5. *If $\{\vec{w} \in W : G(\vec{w}) = (0)\} \subseteq W \backslash \overline{W} = \bigcup_{k=N_i}^{N_{i+1}-1} W_k$, then $d \geq \frac{1}{2}(N_{i+1} - N_i + 1)$.*

Proof. Let Z be the affine variety function of P: if $I \lhd P$ is an ideal, then $Z(I) = \{\vec{w} \in W : I(\vec{w}) = (0)\}$. It's our goal to show that if $Z(GP) \subseteq \bigcup_{k=N_i}^{N_{i+1}-1} W_k$, then $d \geq \frac{1}{2}(N_{i+1} - N_i + 1)$.

Since Q annihilates each W_k, it must annihilate $Z(GP)$ as well. Hilbert's nullstellensatz states that since $\mathbb{K}$ is algebraically closed, for each $q \in Q$, there must be an exponent $q^\pi \in GP$.

Using lemma 2.4, $q^\pi \in R \cap GP \subseteq G + GR$, and so the quotient algebra $R/(G+GR)$ is nil. Since $G^2 \subseteq GR$, R/GR is nil as well. All finitely generated commutative nil algebras are finite dimensional, so applying Lemma 3.2 in [2] several times gives $2d \geq \operatorname{GKdim} R$. Recall that Lemma 3.2 [2] says that if R is a commutative finitely generated graded algebra of Gelfand-Kirillov dimension t, and I is a principal ideal generated by a homogeneous element then R/I has Gelfand-Kirillov dimension at least $t - 1$.

Remember that for any $j \geq 0$, $Q^j = \prod_{k=N_i}^{N_{i-1}-1} (\mathbb{K}x_{k,1} + \mathbb{K}x_{k,2})^{j2^{N_{i+1}-k-1}}$, and:

$$\dim Q^j = \prod_{k=N_i}^{N_{i+1}-1} (j2^{N_{i+1}-k-1} + 1) \geq 2^{\frac{1}{2}(N_{i+1}-N_i-1)(N_{i+1}-N_i)} j^{N_{i+1}-N_i},$$

therefore $\operatorname{GKdim} R \geq N_{i+1} - N_i + 1$. $\square$

We can thus conclude that as long as $\dim F_{i+1} < \frac{1}{2}(N_{i+1} - N_i + 1)$, there is a $\vec{w} \in \overline{W}$ such that $G(\vec{w}) = 0$, and we have appropriate spaces $\{U_k\}$ and monomials $\{v_{k,1}, v_{k,2}\}$ for all $k \leq N_{i+1}$. If this holds for all $i \geq 0$, the induction can proceed.

3 Constructing the ideal E

For any $i \geq 0$, let $V_i = \mathbb{K}v_{i,1} + \mathbb{K}v_{i,2}$, let $v_i \in V_i$ be such that $U_{i+1} = H(2^i)U_i + U_iH(2^i) + v_iH(2^i)$, and let $Q_i = U_i + \mathbb{K}v_i$. If $v_{i,1} \notin \mathbb{K}v_i$, let $W_i = \mathbb{K}v_{i,1}$, otherwise, $W_i = \mathbb{K}v_{i,2}$. This way $Q_i \oplus W_i = H(2^i)$, $U_{i+1} = H(2^i)U_i + Q_iH(2^i)$, and $V_{i+1} = W_iV_i$.

Proposition 3.1. *For any $j > i$ and any $k \leq 2^{j-i} - 1$,*

$$H(k2^i)U_iH(2^j - (k+1)2^i) \subseteq U_j$$

Proof. Apply induction on the value of j by using $H(2^i)U_i + U_iH(2^i) \subseteq U_{i+1}$. □

For any $n > 0$, let $m \geq 0$ be maximal such that $2^m \leq n$, and define:

$$R(n) = \{x \in H(n) : xH(2^{m+1} - n) \subseteq U_{m+1}\}$$

$$L(n) = \{x \in H(n) : H(2^{m+1} - n)x \subseteq U_{m+1}\}$$

Also, set $R(0) = L(0) = (0)$.

Proposition 3.2. *For any $n > 0$ and any M such that $2^M > n$,*

$$R(n)H(2^M - n) \subseteq U_M$$

$$H(2^M - n)L(n) \subseteq U_M$$

Proof. Apply simple induction on M, using the fact that $H(2^M)U_M + U_MH(2^M) \subseteq U_{M+1}$. □

Proposition 3.3. *For any $n > 0$, $R(n)H(1) \subseteq R(n+1)$ and $H(1)L(n) \subseteq L(n+1)$.*

Proof. Let $m \geq 0$ be maxiamal such that $2^m \leq n$. If $2^{m+1} - 1 < n$, then:

$$R(n)H(1) \cdot H(2^{m+1} - n - 1) = R(n)H(2^{m+1} - n) \subseteq U_{m+1},$$

and $R(n)H(1) \subseteq R(n+1)$.

If $2^{m+1} - 1 = n$, then:

$$R(n)H(1) \cdot H(2^{m+2} - n - 1) \subseteq U_{m+1}H(2^{m+1}) \subseteq U_{m+2},$$

and $R(n)H(1) \subseteq R(n+1)$.

By symmetry, $H(1)L(n) \subseteq L(n+1)$. □

Define the space $R'(n) \subseteq H(n)$ recursively; if $n = 0$, set $R(0) = \mathbb{K}$, and otherwise, m be maximal such that $2^m \leq n$ and set:

$$R'(n) = W_m R'(n - 2^m)$$

Note that $\dim R'(n) = 1$.

Proposition 3.4. *For any $n \geq 0$, $R(n) \oplus R'(n) = H(n)$.*

Proof. Use induction on n. The base case $n = 0$ is trivial.

For the inductive step, $n \geq 0$, let m be maximal such that $2^m \leq n$, and assume that $R(n - 2^m) \oplus R'(n - 2^m) = H(n - 2^m)$. Proposition 3.2 can be used to confirm that:

$$Q_m H(n - 2^m) \cdot H(2^{m+1} - n) = Q_m H(2^m) \subseteq U_{m+1},$$

$$H(2^m) R(n - 2^m) \cdot H(2^{m+1} - n) \subseteq H(2^m) U_m \subseteq U_{m+1},$$

$$R(n) + R'(n) \supseteq Q_m H(n - 2^m) + H(2^m) R(n - 2^m) + W_m R'(n - 2^m) = H(n).$$

Since $\dim R'(n) = 1$, either $R(n) \oplus R'(n) = H(n)$ or $R'(n) \subseteq R(n)$. However, the latter option implies $R(n) = H(n)$ and that $H(n) \cdot H(2^{m+1} - n) \subseteq U_{m+1}$, a clear contradiction. Therefore, $R(n) \oplus R'(n) = H(n)$. $\square$

Proposition 3.5. *For any $n \geq 0$,*

$$0 < \dim H(n)/L(n) \leq 2$$

Proof. Let m be maximal such that $2^m \leq n$.

If $H(n)/L(n)$ were zero, then $L(n) = H(n)$ and $H(2^{m+1} - n)H(n) \subseteq U_{m+1}$, a contradiction.

Using proposition 3.2, $R(2^{m+1} - n)H(n) \subseteq U_{m+1}$. By proposition 3.4,

$$L(n) = \{x \in H(n) : R'(2^{m+1} - n)x \in U_{m+1}\}$$

Let $p \in H(2^{m+1} - n)$ be an element that generates $R'(2^{m+1} - n)$, and let $\phi : H(n) \to H(2^{m+1})/U_{m+1}$ be the $\mathbb{K}$-linear transformation:

$$\phi : x \mapsto px/U_{m+1}$$

So that $L(n) = \ker \phi$. Since the image of ϕ is at most dimension 2, $\dim H(n)/L(n) \leq 2$. $\square$

Let $L'(n) \subseteq H(n)$ be a space such that $L(n) \oplus L'(n) = H(n)$. Proposition 3.5 shows that $\dim L'(n)$ is either 1 or 2.

Define the space $E(n) \subseteq H(n)$ as:

$$E(n) = \bigcap_{i=0}^{n} L(i)H(n-i) + H(i)R(n-i)$$

Lemma 3.1. *For any $n > 0$, $E(n)H(1) + H(1)E(n) \subseteq E(n+1)$.*

Proof. Using proposition 3.3,

$$E(n)H(1) = \bigcap_{i=0}^{n} L(i)H(n-i) \cdot H(1) + H(i)R(n-i)H(1) \subseteq$$

$$\bigcap_{i=0}^{n} L(i)H(n+1-i) + H(i)R(n+1-i).$$

It remains to show that $E(n)H(1) \subseteq L(n+1)H(0)+H(n+1)R(0) = L(n+1)$.

Let $m \geq 0$ be maximal such that $2^m \leq n+1$.

$$H(2^{m+1}-n-1)E(n)H(1) \subseteq$$

$$H(2^{m+1}-n-1)L(n-2^m+1)H(2^m) + H(2^m)R(2^m-1)H(1) \subseteq$$

$$U_m H(2^m) + H(2^m)U_m \subseteq U_{m+1}$$

Therefore, by definition, $E(n)H(1) \subseteq L(n+1)$.

$H(1)E(n) \subseteq E(n+1)$ can be proven by symmetry. □

Let $E = \sum_{n=1}^{\infty} E(n)$.

Theorem 3.2. *E is an ideal of A.*

Proof. Apply lemma 3.1 to the definition of E. □

Proposition 3.6. *A/E is infinite dimensional.*

Proof.

$$\dim A/E = \sum_{n=1}^{\infty} \dim H(n)/E(n) > \sum_{n=1}^{\infty} \dim H(n)/R(n) = \sum_{n=1}^{\infty} \dim R'(n) = \infty$$

□

Proposition 3.7. *A/E has quadratic or linear growth.*

Proof. Using the fact that $(L(i)H(n-i)+H(i)R(n-i)) \oplus L'(i)R'(n-i) = H(n)$, and recalling proposition 3.5,

$$\dim H(n)/E(n) \leq \sum_{i=0}^{n} \dim L'(i)R'(n-i) \leq \sum_{i=0}^{n} 2 = 2(n+1),$$

$$\sum_{i=0}^{n} \dim H(i)/E(i) \leq n^2+3n+1.$$

Proposition 3.6 shows algebra isn't finite dimensional. Bergman's Gap Theorem [3] proves that the only growths strictly less than quadratic are linear and finite, so A/E must have quadratic or linear growth. $\square$

4 $E \supseteq \mathcal{E}(F_i)$

Theorem 4.1. *For any $n>0$, let m be maximal such that $2^m \leq n$.*

$$\bigcap_{i=0}^{2^{m+1}-n} \{x \in H(n) : H(i)xH(2^{m+1}-n-i) \subseteq U_m H(2^m) + H(2^m)U_m\} \subseteq E(n).$$

Proof. It's sufficient to show that for any $0 \leq i \leq 2^{m+1}-n$ and any $x \in H(n)$ such that $x \notin L(2^m-i)H(n-2^m+i)+H(2^m-i)R(n-2^m+i)$,

$$H(i)xH(2^{m+1}-n-i) \nsubseteq U_m H(2^m)+H(2^m)U_m.$$

x can be uniquely decomposed into $x_1+x_Lx_R$, with:

$$x_1 \subseteq L(2^m-i)H(n-2^m+i)+H(2^m-i)R(n-2^m+i),$$

$$x_L \subseteq L'(2^m-i),\ x_R \in R'(n-2^m+i)$$

Under our assumption, $x_Lx_R \neq 0$. However,

$$H(i)x_1H(2^{m+1}-n-i) \in$$

$$H(i)L(2^m-i)H(2^m)+H(2^m)R(n-2^m+i)H(2^{m+1}-n-i) \subseteq$$

$$U_m H(2^m) + H(2^m) U_m$$

Therefore it's sufficient to show there exists $y \in H(i)$ and $z \in H(2^{m+1}-n-i)$ such that $yx_L x_R z \notin U_m H(2^m) + H(2^m) U_m$.

As $x_L \notin L(2^m - i)$, there must exist a $y \in H(i)$ such that $yx_L \notin U_m$. Let $yx_L = x_{LU} + x_{LV}$, with $x_{LU} \in U_m$ and $0 \neq x_{LV} \in V_m$. Symmetrically, there's a $z \in H(2^{m+1} - n - i)$ with $x_R = x_{RU} + x_{RV}$, $x_{RU} \in U_m$, and $0 \neq x_{RV} \in V_m$.

$$yx_L x_R z = x_{LU} x_R z + x_{LV} x_{RU} + x_{LV} x_{RV} \notin U_m H(2^m) + H(2^m) U_m$$

□

For any non-zero homogeneous space $F \subseteq H(n)$, let $\mathcal{E}(F)$ denote the space:

$$\mathcal{E}(F) = \bigcap_{j=0}^{n-1} \sum_{k=0}^{\infty} H(kn + j) FA.$$

Proposition 4.1. *For any non-zero homogeneous space $F \subseteq H(n)$, $\mathcal{E}(F)$ is an ideal.*

Proof. By the definition, it's clear that $\mathcal{E}(F)$ is right ideal. To prove it's a left ideal, it's sufficient to show that $H(1)\mathcal{E}(F) \subseteq \mathcal{E}(F)$.

$$H(1)\mathcal{E}(F) = \bigcap_{j=0}^{n-1} \sum_{k=0}^{\infty} H(kn + j + 1) FA =$$

$$\bigcap_{j=1}^{n-1} \sum_{k=0}^{\infty} H(kn + j) FA \cap \sum_{k=0}^{\infty} H(kn + n) FA =$$

$$\bigcap_{j=1}^{n-1} \sum_{k=0}^{\infty} H(kn + j) FA \cap \sum_{k=1}^{\infty} H(kn) FA \subseteq \bigcap_{j=0}^{n-1} \sum_{k=0}^{\infty} H(kn + j) FA = \mathcal{E}(F).$$

□

Corollary 4.2. *For any $i \geq 0$, $\mathcal{E}(F_i) \subseteq E$.*

Proof. Since it's graded, $\mathcal{E}(F_i)$ can decomposed into homogeneous subspaces. If $n < 2^{N_i}$, $\mathcal{E}(F_i) \cap H(n) = \emptyset$, and if $n \geq 2^{N_i}$,

$$\mathcal{E}(F_i) \cap H(n) = \bigcap_{j=0}^{n-1} \sum_{k=0}^{\lfloor (n-j)2^{-N_i}-1 \rfloor} H(k2^{N_i}+j)F_iH(n-(k+1)2^{N_i}-j)$$

Let $n \geq 2^{N_i}$ and let m be maximal such that $2^m \leq n$. For any $0 \leq j \leq 2^{m+1}-n$,

$$H(j)(\mathcal{E}(F_i) \cap H(n))H(2^{m+1}-n-j) \subseteq$$

$$\sum_{k=1}^{\lfloor (n+j)2^{-N_i}-1 \rfloor} H(k2^{N_i})F_iH(2^{m+1}-(k+1)2^{N_i}) \subseteq$$

$$H(k2^{N_i})U_{N_i}H(2^{m+1}-(k+1)2^{N_i}).$$

Using proposition 3.1, this is contained in U_{m+1}, and so by thoerem 4.1, $\mathcal{E}(F_i) \cap H(n) \subseteq E(n)$. □

5 Enumerating elements

To build a Jacobson radical homomorphic image through this method, we use a method very similar to used in Theorem 9 in [7], but readapted for our constraints. First, we require that the field $\mathbb{K}$ be countable, so that we can enumerate the polynomials of $\bar{A}$. For each such $f \in \bar{A}$, we will find a $g \in \bar{A}$ and a sufficiently "small" F such that $f+g-fg \in \mathcal{E}(F)$.

Let $f \subseteq \bar{A}$ be any polynomial with no constant term, and let d be minimal such that $f \in \sum_{n=1}^{d} H(n)$. f can be decomposed as $f = f_{(1)} + \cdots + f_{(d)}$ with each $f_{(i)} \in F(i)$. Recursively define the spaces $s(n) \subseteq H(n)$ for each $n \geq 0$ with:

- $s(0) = 1$,
- $s(n) = \sum_{i=1}^{\min\{n,d\}} f_{(i)}s(n-i)$ for $n > 0$.

This way,

$$s(n) = \sum_{k=0}^{n} \sum_{1 \leq i_1,\ldots,i_k \leq d, i_1+\cdots+i_k=n} f_{(i_1)} \cdots f_{(i_k)}.$$

Lemma 8 from [8] can be used to prove a simple property:

Lemma 5.1. *For any $m_1, m_2 \geq 0$ and any $n \geq m_1 + m_2 + 2d$,*

$$s(n) \subseteq \sum_{a,b=1}^{d} H(m_1 + a)s(n - m_1 - m_2 - a - b + 1)H(m_2 + b - 1)$$

Using s, we can build our subspace F. Recall that $|X|$ is the number of generators of A.

Theorem 5.2. *For any $N \geq 2d$, there exists a homogeneous subspace $F \subseteq H(N)$ with $\dim F \leq \left(\frac{|X|^d-1}{|X|-1}\right)^2$ and a polynomial $g \in \bar{A}$ such that $f+g-fg \in \mathcal{E}(F)$.*

Proof. Let $g = -\sum_{n=1}^{2N+d} s(n)$, and let P be the two-sided ideal generated by $\{s(2N+i)\}_{i=1}^{d}$. By the recursive construction of s,

$$g = -\sum_{n=1}^{2N+d} s(n) = -\sum_{n=1}^{2N+d} \sum_{i=1}^{\min\{n,d\}} f_{(i)} s(n-i) =$$

$$-\sum_{n=1}^{d} f_{(n)} - \sum_{n=1}^{2N+d} \sum_{i=1}^{\min\{n-1,d\}} f_{(i)} s(n-i) = -f - \sum_{i=1}^{d} \sum_{n=i+1}^{2N+d} f_{(i)} s(n-i) =$$

$$-f - \sum_{i=1}^{d} \sum_{n=1}^{2N} f_{(i)} s(n) - \sum_{i=1}^{d} \sum_{n=2N+1}^{2N+d-i} f_{(i)} s(n) \in -f + fg + P$$

Now, set $F = \sum_{a,b=0}^{d-1} H(a)s(N-a-b)H(b)$. It is our goal to show that $P \subseteq \mathcal{E}(F)$. Thanks to proposition 4.1, it sufficient to show that for any $1 \leq i \leq d$, $s(2N+i) \in \mathcal{E}(F)$. Consequently, it's sufficient to show that for any $0 \leq j < N$,

$$s(2N+i) \in H(j)FH(N+i-j) = \sum_{a,b=0}^{d-1} H(j+a)s(N-a-b)H(N+i+b-j),$$

which can be extracted easily from lemma 5.1.

Finally, recall that $\dim H(n) = |X|^n$, where $|X|$ is the number of generators of A.

$$\dim F \leq \sum_{a,b=0}^{d-1} \dim H(a)s(N-a-b)H(b) = \sum_{a,b=0}^{d-1} |X|^{a+b} = \left(\frac{|X|^d - 1}{|X| - 1}\right)^2.$$

$\square$

In order to make our quotient algebra $\bar{A}/E$ Jacobson radical, for every $f \in \bar{A}$ there needs to be a $g \in \bar{A}$ such that $f + g - fg \in E$. As $\bar{A}$ is countable, we can make an enumeration $f_1, f_2, \ldots$. For each f_m, let d_m be minimal such that $f_m \in \sum_{n=1}^{d_m} H(n)$. For any $N_m \geq 1 + \log_2 d_m$, theorem 5.2 can give us a $g_m \in \bar{A}$ and an $F_m \subseteq H(2^{N_m})$ such that $f_m + g_m - f_m g_m \in \mathcal{E}(F_m)$ and $\dim F_m \leq \left(\frac{|X|^{d_m}-1}{|X|-1}\right)^2$.

If each $\dim F_m < \frac{1}{2}(N_m - N_{m-1} + 1)$, then we can construct the ideal E as detailed in section 3. A/E is infinite dimensional (proposition 3.6), has quadratic growth (because affine algebras with linear growth are PI by Small-Stafford-Warfield Theorem [6]) with each $\dim H(n)/E(n) \leq 2(n+1)$ (proposition 3.7), and contains each $\mathcal{E}(F_m)$ (corollary 4.2). Fortunately, each N_m can be set arbitrarily high in relation to N_{m-1}. The needed upper bound of dimension of F_m depends on d_m, $|X|$, N_m and N_{m-1}, so if each N_m is set to $\lceil \sup\{1 + \log_2 d_m, 2\left(\frac{|X|^{d_m}-1}{|X|-1}\right)^2 + N_{m-1}\}\rceil$, each F_m will be "small enough" for the construction of E.

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